A GEOMETRIC INEQUALITY ON HYPERSURFACE IN HYPERBOLIC SPACE

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Abstract. In this paper, we use the inverse curvature flow to prove a sharp geometric inequality on star-shaped and two-convex hypersurface in hyperbolic space.

1. Introduction

The classical Alexandrov-Fenchel inequalities for closed convex hypersurface $\Sigma \subset \mathbb{R}^n$ state that

$$\int_\Sigma \sigma_m(\kappa) d\mu \geq C_{n,m}\Big(\int_\Sigma \sigma_{m-1}(\kappa) d\mu\Big)^{\frac{n-m-1}{n-m}}, \quad 1 \leq m \leq n-1 \tag{1}$$

where $\sigma_m(\kappa)$ is the m-th elementary symmetric polynomial of the principal curvatures $\kappa = (\kappa_1, \cdots, \kappa_{n-1})$ of Σ and $C_{n,m} = \frac{\sigma_m(1,\cdots,1)}{\sigma_{m-1}(1,\cdots,1)}$ is a constant. When $m = 0$, (1) is interpreted as the classical isoperimetric inequality

$$|\Sigma|^{\frac{1}{n-1}} \geq \bar{C}_n Vol(\Omega)^{\frac{1}{n}}, \tag{2}$$

which holds on all bounded domain $\Omega \subset \mathbb{R}^n$ with boundary $\Sigma = \partial\Omega$. Here $|\Sigma|$ is the area Σ and $\bar{C}_n$ is a constant depending only on dimension n. Inequality (1) was generalized to star-shaped and m-convex hypersurface $\Sigma \subset \mathbb{R}^n$ by Guan and Li [8] using the inverse curvature flow recently, where m-convex means that the principal curvature of Σ lies in the Garding's cone

$$\Gamma_m = \{\kappa \in \mathbb{R}^{n-1} | \sigma_i(\kappa) > 0, i = 1, \cdots, m\}.$$

Recently, Huisken [11] showed that in the case $m = 1$, the assumption *star-shaped* can be replaced by *outward-minimizing*.

In this paper, we consider the hyperbolic space $\mathbb{H}^n = \mathbb{R}^+ \times \mathbb{S}^{n-1}$ endowed with the metric

$$\bar{g} = dr^2 + \sinh^2 r g_{\mathbb{S}^{n-1}},$$

where $g_{\mathbb{S}^{n-1}}$ is the standard round metric on the unit sphere $\mathbb{S}^{n-1}$. It's a natural question to establish some analogue inequalities of (1) for closed hypersurface in $\mathbb{H}^n$. In the case of $m = 1$, $\sigma_1 = \sigma_1(\kappa)$ is just the mean curvature H of Σ. Gallego and Solanes [6] have obtained a generalization of (1) to convex hypersurface in hyperbolic space using integral geometric methods,

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however, their result does not seem to be sharp. Denoting $\lambda(r) = \sinh r$, then $\lambda'(r) = \cosh r$. Recently, Brendle,Hung and Wang [3] proved the following inequality for star-shaped and mean convex (i.e.,$H > 0$) hypersurface $\Sigma \subset \mathbb{H}^n$:

$$\int_\Sigma (\lambda' H - (n-1)\langle \bar{\nabla}\lambda', \nu\rangle) d\mu \geq (n-1)\omega_{n-1}^{\frac{1}{n-1}} |\Sigma|^{\frac{n-2}{n-1}} \tag{3}$$

where $|\Sigma|$ is the area of Σ and ω_{n-1} is the area of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. de Lima and Girao [4] also proved the following related inequality independently.

$$\int_\Sigma \lambda' H d\mu \geq (n-1)\omega_{n-1}\left((\frac{|\Sigma|}{\omega_{n-1}})^{\frac{n-2}{n-1}} + (\frac{|\Sigma|}{\omega_{n-1}})^{\frac{n}{n-1}} \right), \tag{4}$$

Both inequalities (3) and (4) are sharp in the sense of that equality holds if and only if Σ is a geodesic sphere centered at the origin. Here, we say a closed hypersurface $\Sigma \subset \mathbb{H}^n$ is star-shaped if the unit outward normal ν satisfies $\langle \nu, \partial_r\rangle > 0$ everywhere on Σ, which is also equivalent to that Σ can be parametrized by a graph

$$\Sigma = \{(r(\theta), \theta) | \theta \in \mathbb{S}^{n-1}\}$$

for some smooth function r on $\mathbb{S}^{n-1}$. We note that inequalities (3) and (4) have some applications in general relativity, see [3, 4, 14].

In this paper, we consider the case $m = 2$. We prove the following sharp inequality for star-shaped and two-convex hypersurface $\Sigma \subset \mathbb{H}^n$, where *two-convex* means that the principal curvature lies in the Garding's cone Γ_2 everywhere on Σ.

Theorem 1. *If $\Sigma \subset \mathbb{H}^n$ is a star-shaped and two-convex hypersurface, then*

$$\int_\Sigma \sigma_2 d\mu \geq \frac{(n-1)(n-2)}{2} \left(\omega_{n-1}^{\frac{2}{n-1}} |\Sigma|^{\frac{n-3}{n-1}} + |\Sigma| \right), \tag{5}$$

where ω_{n-1} is the area of the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ and $|\Sigma|$ is the area of Σ. The equality holds if and only if Σ is a geodesic sphere.

Note that there exists at least one elliptic point on a closed, connected hypersurface Σ in hyperbolic space $\mathbb{H}^n$. Proposition 3.2 in [1] shows that if σ_2 is positive, then σ_1 is automatically positive. So our assumption *two-convex* can also be replaced by $\sigma_2 > 0$ on Σ.

The proof of Theorem 1 follows a similar argument as in [3, 4, 8]. We evolve Σ by a special case of the inverse curvature flow in [7], and consider the following quantity defined by

$$Q(t) = |\Sigma|^{-\frac{n-3}{n-1}} \left(\int_\Sigma \sigma_2 d\mu - \frac{(n-1)(n-2)}{2} |\Sigma| \right).$$

We show that $Q(t)$ is monotone decreasing under the flow. Then we use the convergence result of the flow proved by Gerhardt to estimate a lower

bound of the limit of $Q(t)$:

$$\liminf_{t\to\infty} Q(t) \geq \frac{(n-1)(n-2)}{2}\,\omega_{n-1}^{\frac{2}{n-1}}.$$

In order to estimate this lim inf, we also use a sharp version Sobolev inequality on $\mathbb{S}^{n-1}$ due to Beckner [2] as in [3]. Finally Theorem 1 follows easily from the monotonicity and the lower bound of $\liminf_{t\to\infty} Q(t)$.

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2. Preliminaries

Let $\Sigma \subset \mathbb{H}^n$ be a closed hypersurface with unit outward normal ν. The second fundamental form h of Σ is defined by

$$h(X,Y) = \langle \bar{\nabla}_X \nu, Y\rangle$$

for any two tangent fields X, Y. The principal curvature $\kappa = (\kappa_1, \cdots, \kappa_{n-1})$ are the eigenvalues of h with respect to the induced metric g on Σ. For $1 \leq m \leq n-1$, the m-th elementary symmetric polynomial of κ is defined as

$$\sigma_m(\kappa) = \sum_{i_1 < i_2 \cdots < i_m} \kappa_{i_1} \cdots \kappa_{i_m},$$

which can also be viewed as function of the second fundamental form $h_i^j = g^{jk}h_{ki}$. In the sequel, we will simply write σ_m for $\sigma_m(\kappa)$. We first collect the following basic facts on σ_m (see, e.g, [9,12,13]):

Lemma 2. *Denote* $(T_{m-1})_j^i = \frac{\partial \sigma_m}{\partial h_i^j}$ *and* $(h^2)_i^j = g^{jl}g^{pk}h_{kl}h_{ip}$. *We have*

$$\sum_{i,j} (T_{m-1})_j^i h_i^j = m\sigma_m, \tag{6}$$

$$\sum_{i,j} (T_{m-1})_j^i \delta_i^j = (n-m)\sigma_{m-1} \tag{7}$$

$$\sum_{i,j} (T_{m-1})_j^i (h^2)_i^j = \sigma_1\sigma_m - (m+1)\sigma_{m+1} \tag{8}$$

Moreover, if $\kappa \in \Gamma_m^+$, *we have the following Newton-MacLaurin inequalities*

$$\frac{\sigma_{m-1}\sigma_{m+1}}{\sigma_m^2} \leq \frac{m(n-m-1)}{(m+1)(n-m)} \tag{9}$$

$$\frac{\sigma_1\sigma_{m-1}}{\sigma_m} \geq \frac{m(n-1)}{n-m}, \tag{10}$$

and the equalities hold in (9),(10) *at a given point if and only if* Σ *is umbilical there.*

We now evolve $\Sigma \subset \mathbb{H}^n$ by the following evolution equation

$$\partial_t X = F\nu, \tag{11}$$

where ν is the unit outward normal to $\Sigma_t = X(t, \cdot)$ and F is the speed function which may depend on the position vector, principal curvatures and time. Let g_{ij} be the induced metric and $d\mu_t$ be its area element on Σ_t. We have the following evolution equations.

Proposition 3. *Under the flow* (11)*, we have:*

$$\begin{aligned}
&\partial_t g_{ij} = 2Fh_{ij}\\
&\partial_t \nu = -\nabla F,\\
&\partial_t h_i^j = -\nabla^j\nabla_i F - F(h^2)_i^j + F\delta_i^j,\\
&\partial_t d\mu = F\sigma_1 d\mu, \qquad\qquad (12)\\
&\partial_t \sigma_m = -\nabla^i((T_{m-1})_i^j\nabla_j F) - F(\sigma_1\sigma_m - (m+1)\sigma_{m+1})\\
&\qquad\quad + (n-m)F\sigma_{m-1}, \qquad\qquad (13)
\end{aligned}$$

Proof. The first four equations follow from direct computations like in [10]. Now we calculate the evolution of σ_m (cf. [8])

$$\begin{aligned}
\partial_t\sigma_m =&\frac{\partial\sigma_m}{\partial h_i^j}\partial_t h_i^j\\
=&-(T_{m-1})_j^i\nabla^j\nabla_i F - F(T_{m-1})_j^i(h^2)_i^j + F(T_{m-1})_j^i\delta_i^j\\
=&-\nabla^j((T_{m-1})_j^i\nabla_i F) - F(\sigma_1\sigma_m - (m+1)\sigma_{m+1})\\
&\quad + (n-m)F\sigma_{m-1},
\end{aligned}$$

where in the last equality we used (7),(8) and the divergence free property of $(T_{m-1})_j^i$ (see [13]). ☐

Proposition 4. *Under the flow* (11)*, we have*

$$\frac{d}{dt}\int_\Sigma \sigma_m d\mu =(m+1)\int_\Sigma F\sigma_{m+1}d\mu + (n-m)\int_\Sigma F\sigma_{m-1}d\mu.$$

Proof. This proposition follows directly from (12), (13) and the divergence theorem. ☐

In [7] Gerhardt studied general inverse curvature flow of star-shaped hypersurface in hyperbolic space. For our purpose, we will use a special case of their result for the following flow

$$\partial_t X = \frac{n-2}{2(n-1)}\frac{\sigma_1}{\sigma_2}\nu. \tag{14}$$

Theorem 5 (Gerhardt [7])**.** *If the initial hypersurface is star-shaped and strictly two-convex, then the solution for the flow* (14) *exists for all time* $t > 0$ *and the flow hypersurfaces converge to infinity while maintaining star-shapedness and strictly two-convex. Moreover, the hypersurfaces become*

strictly convex exponentially fast and more and more totally umbilical in the sense of

$$|h_i^j - \delta_i^j| \leq Ce^{-\frac{t}{n-1}}, \quad t > 0,$$

i.e., the principal curvatures are uniformly bounded and converge exponentially fast to one.

3. Monotonicity

We define the quantity

$$Q(t) = |\Sigma_t|^{-\frac{n-3}{n-1}} \left(\int_{\Sigma_t} \sigma_2 d\mu - \frac{(n-1)(n-2)}{2} |\Sigma_t| \right),$$

where $|\Sigma_t|$ is the area of Σ_t. In this section, we show that $Q(t)$ is monotone decreasing along the flow (14).

Proposition 6. *Under the flow* (14)*, the quantity $Q(t)$ is monotone decreasing. Moreover, $\frac{d}{dt}Q(t) = 0$ at some time t if and only if the surface Σ_t is totally umbilical.*

Proof. Under the flow (14), Proposition 4 and (12) imply that

$$\frac{d}{dt} \int_{\Sigma} \sigma_2 d\mu = \frac{3(n-2)}{2(n-1)} \int_{\Sigma} \frac{\sigma_1 \sigma_3}{\sigma_2} d\mu + \frac{(n-2)^2}{2(n-1)} \int_{\Sigma} \frac{\sigma_1^2}{\sigma_2} d\mu \tag{15}$$

$$\frac{d}{dt} |\Sigma_t| = \frac{(n-2)}{2(n-1)} \int_{\Sigma} \frac{\sigma_1^2}{\sigma_2} d\mu. \tag{16}$$

Combining (15), (16) and (9), we have

$$\frac{d}{dt} \left(\int_{\Sigma} \sigma_2 d\mu - (n-2)|\Sigma_t| \right) \leq \frac{n-3}{n-1} \int_{\Sigma} \sigma_2 d\mu. \tag{17}$$

By applying the Newton-MacLaurin inequality (10) in (16), we also have

$$\frac{d}{dt} |\Sigma_t| \geq |\Sigma_t|. \tag{18}$$

Then combining (17) and (18) gives that

$$\frac{d}{dt} \left(\int_{\Sigma} \sigma_2 d\mu - \frac{(n-1)(n-2)}{2} |\Sigma_t| \right) \leq \frac{n-3}{n-1} \left(\int_{\Sigma} \sigma_2 d\mu - \frac{(n-1)(n-2)}{2} |\Sigma_t| \right). \tag{19}$$

From Proposition 8 in the next section and (19), we know that the quantity

$$\int_{\Sigma} \sigma_2 d\mu - \frac{(n-1)(n-2)}{2} |\Sigma_t|$$

is nonnegative along the flow (14). Then inequalities (18) and (19) implies that

$$\frac{d}{dt} Q(t) \leq 0.$$

If the equality holds, the inequalities (9) and (10) assume equalities everywhere on Σ_t. Then Σ_t is totally umbilical. $\square$

4. The asymptotic behavior of monotone quantity

In this section, we use the convergence result of the flow (14) proved in [7] to estimate the lower bound of the limit of $Q(t)$. First we need the following sharp Sobolev inequality on $\mathbb{S}^{n-1}$ ([2]).

Lemma 7. *For every positive function f on $\mathbb{S}^{n-1}$, we have*

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} f^{n-3} dvol_{\mathbb{S}^{n-1}} + \frac{n-3}{n-1}\int_{\mathbb{S}^{n-1}} f^{n-5}|\nabla f|^2 dvol_{\mathbb{S}^{n-1}}\\ \geq \omega_{n-1}^{\frac{2}{n-1}}\Big(\int_{\mathbb{S}^{n-1}} f^{n-1} dvol_{\mathbb{S}^{n-1}}\Big)^{\frac{n-3}{n-1}}.\end{aligned}$$

Moreover, equality holds if and only if f is a constant.

Proof. From Theorem 4 in [2], for any positive smooth function w on $\mathbb{S}^{n-1}$, we have the following inequalty

$$\begin{aligned}\frac{4}{(n-1)(n-3)}\int_{\mathbb{S}^{n-1}} |\nabla w|^2 dvol_{\mathbb{S}^{n-1}} + \int_{\mathbb{S}^{n-1}} w^2 dvol_{\mathbb{S}^{n-1}}\\ \geq \omega_{n-1}^{\frac{2}{n-1}}\Big(\int_{\mathbb{S}^{n-1}} w^{\frac{2(n-1)}{n-3}} dvol_{\mathbb{S}^{n-1}}\Big)^{\frac{n-3}{n-1}}.\end{aligned}$$

Moreover equality holds if and only if w is constant. For any positive function f on $\mathbb{S}^{n-1}$, by letting $w = f^{\frac{n-3}{2}}$, we have

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} f^{n-3} dvol_{\mathbb{S}^{n-1}} + \frac{n-3}{n-1}\int_{\mathbb{S}^{n-1}} f^{n-5}|\nabla f|^2 dvol_{\mathbb{S}^{n-1}}\\ \geq \omega_{n-1}^{\frac{2}{n-1}}\Big(\int_{\mathbb{S}^{n-1}} f^{n-1} dvol_{\mathbb{S}^{n-1}}\Big)^{\frac{n-3}{n-1}}\end{aligned}$$

and equality holds if and only if f is a constant. $\square$

Proposition 8. *Under the flow* (14)*, we have*

$$\liminf_{t\to\infty} Q(t) \geq \frac{(n-1)(n-2)}{2}\ \omega_{n-1}^{\frac{2}{n-1}}. \tag{20}$$

Proof. Recall that star-shaped hypersurfaces can be written as graphs of function $r = r(t,\theta), \theta \in \mathbb{S}^{n-1}$. Denote $\lambda(r) = \sinh(r)$, then $\lambda'(r) = \cosh(r)$. We next define a function φ on $\mathbb{S}^{n-1}$ by $\varphi(\theta) = \Phi(r(\theta))$, where $\Phi(r)$ is a positive function satisfying $\Phi' = 1/\lambda$. Let $\theta = \{\theta^j\}, j = 1, \cdots, n-1$ be a coordinate system on $\mathbb{S}^{n-1}$ and φ_i, φ_{ij} be the covariant derivatives of φ with respect to the metric $g_{\mathbb{S}^{n-1}}$. Define

$$v = \sqrt{1 + |\nabla\varphi|^2_{\mathbb{S}^{n-1}}}.$$

From [7], we know that

$$\lambda = O(e^{\frac{t}{n-1}}), \qquad |\nabla\varphi|_{\mathbb{S}^{n-1}} + |\nabla^2\varphi|_{\mathbb{S}^{n-1}} = O(e^{-\frac{t}{n-1}}) \tag{21}$$

Since $\lambda' = \sqrt{1+\lambda^2}$, we have

$$\lambda' = \lambda(1 + \frac{1}{2}\lambda^{-2} + O(e^{-\frac{4t}{n-1}})) \tag{22}$$

From (21) we also have

$$\frac{1}{v} = 1 - \frac{1}{2}|\nabla\varphi|^2_{\mathbb{S}^{n-1}} + O(e^{-\frac{4t}{n-1}}) \tag{23}$$

In terms of φ, we can express the metric and second fundamental form of Σ as following (see, e.g, [3, 5])

$$\begin{aligned} g_{ij} =& \lambda^2(\sigma_{ij} + \varphi_i\varphi_j) \\ h_{ij} =& \frac{\lambda'}{v\lambda} g_{ij} - \frac{\lambda}{v}\varphi_{ij}, \end{aligned}$$

where $\sigma_{ij} = g_{\mathbb{S}^{n-1}}(\partial_{\theta^i}, \partial_{\theta^j})$. Denote $a_i = \sum_k \sigma^{ik}\varphi_{ki}$ and note that $\sum_i a_i = \Delta_{\mathbb{S}^{n-1}}\varphi$. By (21), the principal curvatures of Σ_t has the following form

$$\kappa_i = \frac{\lambda'}{v\lambda} - \frac{a_i}{v\lambda} + O(e^{-\frac{4t}{n-1}}), \quad i = 1, \cdots, n-1.$$

Then we have

$$\begin{aligned} \sigma_2 =& \sum_{i<j} \kappa_i\kappa_j \\ =& \frac{(n-1)(n-2)}{2}(\frac{\lambda'}{v\lambda})^2 - (n-2)\frac{\lambda'\Delta_{\mathbb{S}^{n-1}}\varphi}{v^2\lambda^2} + O(e^{-\frac{4t}{n-1}}). \end{aligned}$$

By using (22) and (23),

$$\begin{aligned} \sigma_2 =& \frac{(n-1)(n-2)}{2}(1 + \lambda^{-2} - |\nabla\varphi|^2_{\mathbb{S}^{n-1}}) \\ & - (n-2)\lambda^{-1}\Delta_{\mathbb{S}^{n-1}}\varphi + O(e^{-\frac{4t}{n-1}}). \end{aligned}$$

On the other hand,

$$\sqrt{\det g} = (\lambda^{n-3} + O(e^{\frac{(n-3)t}{n-1}}))\sqrt{\det g_{\mathbb{S}^{n-1}}}.$$

So we have

$$
\begin{aligned}
&\int_{\Sigma_t}\sigma_2 d\mu-\frac{(n-1)(n-2)}{2}|\Sigma_t|\\
=&\int_{\mathbb{S}^{n-1}}\lambda^{n-1}(\sigma_2-\frac{(n-1)(n-2)}{2})dvol_{\mathbb{S}^{n-1}}+O(e^{\frac{(n-5)t}{n-1}})\\
=&\frac{(n-1)(n-2)}{2}\int_{\mathbb{S}^{n-1}}(\lambda^{n-3}-\lambda^{n-1}|\nabla\varphi|^2_{\mathbb{S}^{n-1}})dvol_{\mathbb{S}^{n-1}}\\
&\qquad-(n-2)\int_{\mathbb{S}^{n-1}}\lambda^{n-2}\Delta_{\mathbb{S}^{n-1}}\varphi dvol_{\mathbb{S}^{n-1}}+O(e^{\frac{(n-5)t}{n-1}})\\
=&\frac{(n-1)(n-2)}{2}\int_{\mathbb{S}^{n-1}}(\lambda^{n-3}-\lambda^{n-1}|\nabla\varphi|^2_{\mathbb{S}^{n-1}})dvol_{\mathbb{S}^{n-1}}\\
&\qquad+(n-2)^2\int_{\mathbb{S}^{n-1}}\lambda^{n-3}\nabla\lambda\nabla\varphi dvol_{\mathbb{S}^{n-1}}+O(e^{\frac{(n-5)t}{n-1}}).
\end{aligned}
$$

Since $\nabla\lambda=\lambda\lambda'\nabla\varphi$, by using (22), we deduce that

$$
\begin{aligned}
&\int_{\Sigma_t}\sigma_2 d\mu-\frac{(n-1)(n-2)}{2}|\Sigma_t|\\
=&\frac{(n-1)(n-2)}{2}\int_{\mathbb{S}^{n-1}}(\lambda^{n-3}+\frac{n-3}{n-1}\lambda^{n-5}|\nabla\lambda|^2)dvol_{\mathbb{S}^{n-1}}+O(e^{\frac{(n-5)t}{n-1}}).
\end{aligned}
\tag{24}
$$

Moreover,

$$
|\Sigma_t|^{\frac{n-3}{n-1}}=(\int_{\mathbb{S}^{n-1}}\lambda^{n-1}dvol_{\mathbb{S}^{n-1}})^{\frac{n-3}{n-1}}+O(e^{\frac{(n-5)t}{n-1}}). \tag{25}
$$

Using Lemma 7, we can complete the proof of Proposition 8 by combining (24) and (25). $\square$

We now complete the proof of Theorem 1

Proof of Theorem 1. Since $Q(t)$ is monotone decreasing, we have

$$
Q(0)\geq\liminf_{t\to\infty}Q(t)\geq\frac{(n-1)(n-2)}{2}\ \omega_{n-1}^{\frac{2}{n-1}}.
$$

This gives that the initial hypersurface Σ satisfies

$$
\left(\int_{\Sigma}\sigma_2 d\mu-\frac{(n-1)(n-2)}{2}|\Sigma|\right)\geq\frac{(n-1)(n-2)}{2}\ \omega_{n-1}^{\frac{2}{n-1}}|\Sigma|^{\frac{n-3}{n-1}},
$$

which is equivalent to the inequality (5) in Theorem 1. Now we assume that equality holds in (5), which implies that $Q(t)$ is a constant. Then Proposition 6 implies Σ_t is umbilical and therefore a geodesic sphere. It is also easy to see that if Σ is a geodesic sphere of radius r, then the area of Σ

is $|\Sigma| = \omega_{n-1}\sinh^{n-1} r$ and the integral of σ_2 is

$$\begin{aligned}\int_\Sigma \sigma_2 d\mu =&\frac{(n-1)(n-2)}{2}\omega_{n-1}\coth^2 r\sinh^{n-1} r\\ =&\frac{(n-1)(n-2)}{2}\omega_{n-1}\left(\sinh^{n-1} r+\sinh^{n-3} r\right)\\ =&\frac{(n-1)(n-2)}{2}\left(|\Sigma|+\omega_{n-1}^{\frac{2}{n-1}}|\Sigma|^{\frac{n-3}{n-1}}\right).\end{aligned}$$

Hence the equality holds in (5) on a geodesic sphere. This completes the proof of Theorem 1. □

References

[1] J. L. M. Barcosa and A. G. Colares, *Stability of Hypersurfaces with Constant r-Mean Curvature*, Annals of Global Analysis and Geometry **15**(1997), 277-297.

[2] W. Beckner, *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality,* Ann. of Math. **138**(1993), 213-242.

[3] S. Brendle, P.-K. Hung, and M.-T. Wang, *A Minkowski-type inequality for hypersurfaces in the Anti-deSitter-Schwarzschild manifold*, arXiv: 1209.0669.

[4] L.L. de lima and F. Girão, *An Alexandrov-Fenchel-type inequality in hyperbolic space with an application to a penrose inequality*, arXiv: 1209.0438.

[5] Q. Ding, *The inverse mean curvature flow in rotationally symmetric spaces*, Chinese Annals of Mathematics, Series B, 1-18 (2010)

[6] E. Gallego and G. Solanes, *Integral geometry and geometric inequalities in hyperbolic space*, Differential Geom. Appl. **22**(2005), 315-325

[7] C. Gerhardt, *Inverse curvature flows in hyperbolic space*, J. Differential Geom. **89** (2011), no. 3, 487-527.

[8] P. Guan and J. Li, *The quermassintegral inequalities for k-convex starshaped domains,*Adv. Math. **221**(2009), 1725-1732.

[9] G.H. Hardy, J.E. Littlewood, G. Polya, *Inequalities*, Cambridge Univ. Press, Cambridge, 1934.

[10] G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Differential Geom. **20** (1984) 237-266.

[11] G. Huisken, in preparation

[12] G. Huisken and C. Sinestrari, *Convexity estimates for mean curvature flow and singularities of mean convex surfaces,* Acta Math. **183** (1999) 45-70.

[13] R. Reilly, *On the Hessian of a function and the curvatures of its graph*, Michigan Math. J. **20** (1973) 373-383.

[14] M.-T. Wang, *Quasilocal mass and surface Hamiltonian in spacetime*, arXiv:1211.1407

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